

Topic of this homework: Visualizing complex functions; Bilinear/Möbius transform; Riemann sphere, Branch cuts; Fourier and Laplace transforms.

1 Visualizing complex functions

The mapping from $z = x + iy$ to $w(z) = u(x, y) + iv(x, y)$ is a $2 \cdot 2 = 4$ dimensional graph. This is difficult to visualize, because for each point in the domain z , we would like to represent both the magnitude and phase (or real and imaginary parts) of $w(z)$. A good way to visualize these mappings is to use color to represent the phase and hue (dark to light) to represent the magnitude.¹ The Matlab program `zviz.m` has been provided to do this (see Lecture 18 on the class website: <http://web.engr.illinois.edu/~jontalle/uploads/298/zviz.m>)

To use the program in Matlab, use the syntax `zviz <function of z>` (for example, type `zviz z^2`). This will render a colorized version of the function. Examples you can render with `zviz` are given in the comments at the top of the `zviz.m` program. You can view the colormap (how color and hue related to magnitude and phase) by running `zviz z`.

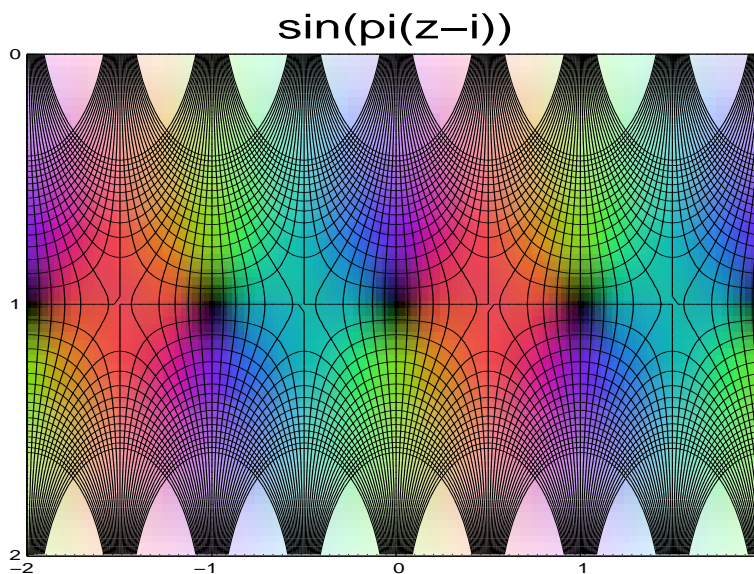


Figure 1: *Plot of $\sin(\pi(z - i))$.*

Example: $\sin(\pi(z - i))$ is rendered as shown in Fig. 1 by the command `zviz sin(pi*(z-1i))`. The abscissa (horizontal axis) is the real x axis and the ordinate (vertical axis) is the complex iy axis. The graph is offset along the iy axis by $1i$ since the argument $z - i$ causes a shift of the sine function by 1 in the positive imaginary direction. The zeros of the sine function occur at integer multiples of $z = \pi$, so they appear as dark regions on the horizontal line $y = 1$. Along this horizontal axis the function oscillates between red (phase is zero or a multiple of 2π , meaning the function is positive) and blue (phase is π or an odd multiple of π , meaning the function is negative, because $e^{i\pi} = -1$). Along the vertical axis, the function is either a $\cosh(y)$ or $\sinh(y)$; the hue becomes very light as $|y|$ increases, indicating that the magnitude of the function goes to infinity. The black grid lines are contour lines for the real and imaginary parts of the function. They show how the rectangular $z = x + yj$ contour lines (axes) have been transformed (mapped) by the function.

¹This is sometimes called ‘domain coloring’: https://en.wikipedia.org/wiki/Domain_coloring

To do: For the following functions, explain what you see (i.e., understand) about the **zviz** plot of each function. Do not include printouts of the plots with your homework. Write down your observations, in prose.

1. z (How does the color relate to the phase of z ? How does the hue relate to the magnitude of z ?)
2. z^2
3. e^z
4. $\cos(\pi z/2)$
5. $\cosh(\pi z)$

2 Fundamental theorem of algebra (FTA) vs. Bézout's theorem

1. State the fundamental theorem of algebra (FTA).
2. State Bézout's theorem.
3. Compare and contrast these two theorems. Is there a relationship between them?

3 Möbius transforms and infinity

The bilinear transform: The *bilinear z transform* (a specific case of the Möbius transformation) is used in signal processing to design a digital (discrete-time) filter $H(z)$ given an analog (continuous time) filter $H(s)$. The goal of the transform is to take a function of analog frequency ω_a , where $\omega_a \in (-\infty, \infty)$, and map it to a finite digital frequency range, $\omega_d \in [-\pi, \pi]$. You will learn more about this if you take ECE 310.

The bilinear z transform is expressed in terms of the complex Laplace frequency $s \equiv \sigma_a + j\omega_a$, where ω_a is the analog frequency in radians/second, and $z \equiv e^{\sigma_d + j\omega_d}$, where ω_d is the digital frequency in radians. The bilinear transform is given by

$$s = \alpha \frac{1 - z^{-1}}{1 + z^{-1}}, \quad (1)$$

where α is a real constant.

1. Suppose you are given the analogue low-pass filter $h(t) = e^{-t}u(t)$, which has a Laplace transform of

$$H(s) = \frac{1}{s + 1} = \int_{0^-}^{\infty} h(t)e^{-st}dt.$$

Use the bilinear z transform (Eq. 1) to find the discrete time filter $H(z)$. *Hint: Your answer should be a composition of $H(s)$ and Eq. 1.*

2. Substitute $s = j\omega_a$ and $z = e^{j\omega_d}$ ($\sigma_a, \sigma_d = 0$) into the Eq. 1 to determine the relationship between ω_a , ω_d and α . Express your final result using a tangent function. *Hint: Try to form sine and cosine terms! Recall that $\sin(\omega) = (e^{j\omega} - e^{-j\omega})/2j$ and $\cos(\omega) = (e^{j\omega} + e^{-j\omega})/2$.*
3. By hand, draw a graph of the relationship you found the previous part, $\omega_a = f(\omega_d)$. Make sure to specify the behavior of ω_a at $\omega_d = 0, \pm\pi/2, \pm\pi$.
4. Explain how this relationship maps the analog frequency $\omega_a \rightarrow \pm\infty$ to the digital frequency ω_d .

5. Draw the s and z planes, showing the real parts on the horizontal axes and the imaginary parts on the vertical axes. Mark (e.g. using thick lines) which sets of values are considered when $\sigma_a, \sigma_d = 0$.
6. Geometrically, what is the effect of this Möbius transform? Consider your drawing in the previous part.

4 Fourier and Laplace Transforms

In this problem we review key similarities and differences of Fourier and Laplace transforms.

Basic definitions: The *Dirac delta function* $\delta(t)$ is not a true function, rather it is defined under an integral sign as

$$u(t) = \int_{-\infty}^t \delta(t) dt.$$

From the *Fundamental theorem of calculus*, it follows that

$$\delta(t) \equiv \frac{d}{dt} u(t).$$

When the Dirac delta is multiplied by a function under an integral, the result of the integral is the value of the function where the argument of the delta function goes to zero:

$$f(nT_s) = \int_{-\infty}^{\infty} f(t) \delta(t - nT_s) dt.$$

In digital signal processing this is called *sampling* $f(t)$ at $t_n = nT_s$, where $t_n, T_s \in \mathbb{R}$ and $n \in \mathbb{Z}$. Sampling converts the “analogue” signal $f(t)$ into a “digital” (discrete time) signal $f(nT_s)$. The constant T_s is called the *Nyquist sample period*.

The *Heaviside step function* is defined as

$$u(t) = \int_{-\infty}^t \delta(t) dt = \begin{cases} 1 & \text{if } t > 0 \\ \text{Not Defined} & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The *Fourier step function* is defined differently as

$$\tilde{u}(t) \equiv \frac{1 + \text{sgn}(t)}{2} = \begin{cases} 1 & \text{if } t > 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$

We can calculate the FT of this step function by taking advantage of linearity property of the FT.² The FT of $\tilde{u}(t)$ is calculable because the FTs of 1 and $\text{sgn}(t)$ are both calculable. Note that for our purposes here, the sign function $\text{sgn}(t)$ is defined as

$$\text{sgn}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

²FTs and LTs are linear transforms, meaning they obey the superposition property: If $f_1 \leftrightarrow F_1$ and $f_2 \leftrightarrow F_2$, by linearity $\alpha f_1 + \beta f_2 \leftrightarrow \alpha F_1 + \beta F_2$.

To Do:

1. Fourier transforms (FTs)

- (a) Give the FTs (you may look them up) of $f(t) = 1$ and $f(t) = \text{sgn}(t)$.
- (b) Use the FTs of 1 and $\text{sgn}(t)$ to find the FT of $\tilde{u}(t)$.

2. Laplace transforms (LTs)

- (a) By applying the definition of the LT (Lec 20), find $U(s) \leftrightarrow u(t)$. That is, evaluate the integral

$$U(s) = \int_{0^-}^{\infty} u(t)e^{-st} dt.$$

Hint: You must assume the real part of s is positive ($\Re\{s\} > 0$)

- (b) Find $\Delta(s) \leftrightarrow \delta(t - T_0)$, defined as

$$\Delta(s) = \int_{0^-}^{\infty} \delta(t - T_0)e^{-st} dt.$$

- (c) The definition of the LT has 0^- as the lower limit of the integral. Explain what this means, and why it is necessary.
- (d) Evaluate the convolution $u(t) \star u(t)$ (i.e., do the integral).

3. Compare Fourier and Laplace transforms (FTs vs. LTs):

Fill out the table of some basic Fourier and Laplace transforms. Here we define $f(t) \leftrightarrow F(\omega)$ as a *Fourier transform* and $f(t)u(t) \leftrightarrow F(s)$ as a *Laplace transform*, where $s = \sigma + j\omega$ is the Laplace radian frequency. If the transform does not exist, write ‘DNE.’

$f(t)$	\leftrightarrow	$F(\omega)$	$F(s)$
$\delta(t - T_0)$			
1			
$\text{sgn}(t)$			
$\tilde{u}(t)$			
$u(t)$			
$tu(t) = u(t) \star u(t)$			
$e^{-at}u(t)$			